The Discontinuous Galerkin Method on Dynamical hp-Meshes

Sascha M. Schnepp[1] and Thomas Weiland[2]

[1] Graduate School Computational Engineering, Technische Universitaet Darmstadt, Dolivostrasse 15, 64293 Darmstadt, Germany schnepp@gsc.tu-darmstadt.de,

[2] Institut fuer Theorie Elektromagnetischer Felder, Technische Universitaet Darmstadt, Schlossgartenstrasse 8, 64289 Darmstadt, Germany thomas.weiland@temf.tu-darmstadt.de

Summary. An hp-adaptive Discontinuous Galerkin method for time-domain electromagnetics problems is proposed. The method allows for arbitrary anisotropic refinements in the approximation order p and the mesh step size h regardless of the resulting level of hanging nodes. The adaptation process is guided by so-called reference solutions [14, 15], which are employed for estimating the solution error and finding the best type of refinement.

1 Introduction

In this article, we are concerned with solving the Maxwell equations for electromagnetic fields with arbitrary time dependence in a three-dimensional domain $\Omega \subset \mathbb{R}^3$. They read

$$\nabla \times \mathbf{E}(\mathbf{x},t) = -\frac{\partial}{\partial t}\mu(\mathbf{x})\mathbf{H}(\mathbf{x},t), \tag{1a}$$

$$\nabla \times \mathbf{H}(\mathbf{x},t) = \frac{\partial}{\partial t}\varepsilon(\mathbf{x})\mathbf{E}(\mathbf{x},t) + \mathbf{J}(\mathbf{x},t), \tag{1b}$$

with the spatial variable $\mathbf{x} \in \Omega$ and the temporal variable $t \in [t_0, T] \subset \mathbb{R}$ subject to boundary conditions specified at the domain boundary $\partial\Omega$ and initial conditions specified at time t_0. The electric and magnetic field vectors are denoted by $\mathbf{E}$ and $\mathbf{H}$, $\mathbf{J}$ denotes the electric current density. In (1), we assumed resting heterogeneous, linear, isotropic, non-dispersive and time-independent materials. The magnetic permeability and dielectric permittivity μ and ε for this case are scalar values depending on the spatial position only.

For discretizing (1), we employ the discontinuous Galerkin (DG) method [1, 2]. Nowadays, the DG method has gained wide acceptance as a numerical method, which combines the key features of accuracy and flexibility. Its flexibility stems from the highly localized character of the numerical approximation. This renders the method specially suited for time-domain problems as well as for applying adaptive mesh refinement. In particular, the method can easily deal with meshes with hanging nodes as stated in [3], which makes it particularly well suited for hp-adaptivity, i.e., the adaptation of the computational mesh regarding the local mesh step size h and the local approximation order p.

There is a well established body of literature on the DG method for various types of problems available. It has been thoroughly investigated by several research groups (see e.g. [3–5] and references therein). Concerning Maxwell's equations in time-domain, the DGM has been studied in particular in [5–8].

This paper focuses on error controlled dynamic hp-adaptation. In parts, it is a continuation of our work in [13], where a general formulation of the DGM on non-regular hexahedral meshes was introduced. The first published work on h-, p- and hp-adaptivity within the DG framework is presumably [9], where the authors consider linear scalar hyperbolic conservation laws in two dimensional space. For a selection of other publications see [10–12] and the references therein. Our formulation allows for arbitrary anisotropic h- and p-refinements with very relaxed demands on the level of hanging nodes.

2 Automatic and dynamic mesh adaptation with the DG method

The space and time continuous electromagnetic fields are approximated on a tesselation $\mathscr{T}$ of the domain of interest Ω. The approximation of the electric field local to the element with index i reads

$$\mathbf{E}_i(\mathbf{x},t) = \sum_p \mathbf{e}_i^p(t)\varphi_i^p(\mathbf{x}), \quad \mathbf{x} \in \mathscr{T}_i \tag{2}$$

with the polynomial basis functions $\varphi(\mathbf{x})$ of order $p \in \mathscr{P} = \{0, .., P\}$ and the time-dependent vector of coefficients $\mathbf{e}$. The magnetic field is approximated respectively.

It is specific to the DG method that the basis functions are defined with element-wise compact support. As a consequence the individual element-local approximations are not trivially connected, which inherently leads to a globally discontinuous approximation. Element communication is established via the so-called numerical interface fluxes only, which appear in the form of element surface integrals in the weak formulation of (1) (see e.g. [13] for details). This high degree of localization turns mesh adaptation into a purely element-local operation.

By defining proper finite element spaces associated with refined or reduced elements of h-, p- and hp-type the best approximation in the L^2-sense, f^*, of

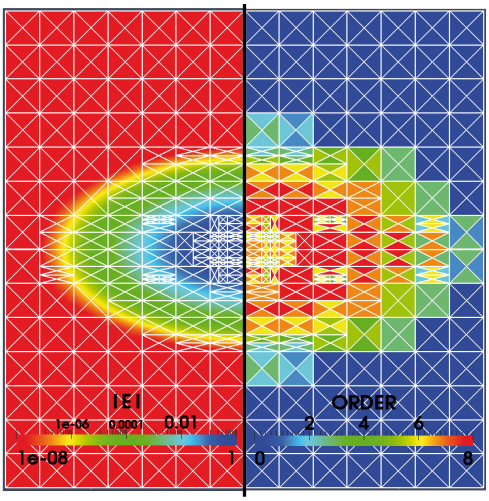

Fig. 1. Cut view of the electric field magnitude having a 3D Gaussian distribution (left) and corresponding anisotropically refined *hp*-mesh for an error tolerance of 10^{-9} in the L^2 norm. The mesh view makes use of the common tensor product order visualization technique [14].

a DG approximation f given on an existing *hp*-mesh is obtained by the orthogonal projection operator Π^p

$$f^* = \sum_p \Pi^p(f)_{\mathscr{T}_i} \varphi_i^p = \sum_p \frac{\left(\varphi_i^p, f\right)_{\mathscr{T}_i}}{\left(\varphi_i^p, \varphi_i^p\right)_{\mathscr{T}_i}} \varphi_i^p, \quad (3)$$

where $(u,v)_{\mathscr{T}_i}$ denotes the inner product $\int_{\mathscr{T}_i} uv\,\mathrm{d}\mathbf{x}$ on the element $\mathscr{T}_i$. In [13] it is shown that this projection can be performed very efficiently, and that it guarantees stability by respecting the electromagnetic energy of the current field solution as a strict upper limit.

In order to perform automatic mesh adaptation, the approximation error has to be estimated in an element-wise fashion in a first step. In [13] an error estimator based the size of the interelement jumps of the DG solution was proposed. In a second step the best type of adaptation, i.e., *h*-, *p*- and *hp*-refinement and/or reduction, has to be determined. This information is inferred from a local regularity estimation.

In this contribution, we apply the concept of *reference solutions* [14, 15]. A reference solution is a numerically computed approximation, which is assumed to be significant more accurate than the present approximation. This can be achieved by performing one isotropic *h*-refinement combined with increasing the approximation order by one on the element under consideration. The error of the present solution is computed with respect to the reference solution, which is also employed for finding the best refinement out of a list of candidates. Figure 1 shows an example of an anisotropically refined *hp*-mesh yielding an approximation error below 10^{-9} in the L^2-norm. We adopted the common tensor product order visualization technique of [14, 15], where the color of a triangle including, e.g., an x-directed edge visualizes the order P_x.

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