Adaptive-order rational Arnoldi method for Maxwell's equations

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Summary. We present some new results for model order reduction of Maxwell's equations using an adaptive-order rational Arnoldi method. In this context, we introduce a new adaptive choice of expansion points.

1 Introduction

In view of the increasing frequency range and the progressing miniaturization, the analysis of parasitic effects has become an important task for the development of integrated circuits. The appearing phenomena, e.g. crosstalk or signal delay, are usually modelled using Maxwell's equations.
Since high-dimensional model problems are often necessary for accurate simulations, model order reduction techniques are an important tool for the proper and fast analysis of these phenomena.

1.1 Model Order Reduction

We will apply model order reduction for linear time-invariant descriptor systems

$$\begin{aligned} \mathscr{E}\dot{x}(t) &= \mathscr{A}x(t) + \mathscr{B}u(t), \\ y(t) &= \mathscr{C}x(t), \end{aligned} \tag{1}$$

where $\mathscr{E}, \mathscr{A} \in \mathbb{R}^{N\times N}$, $\mathscr{B} \in \mathbb{R}^{N\times m}$ and $\mathscr{C} \in \mathbb{R}^{p\times N}$. Furthermore, $u(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^p$ denote the input and the output of the descriptor system, respectively. In general, descriptor systems are associated with the transfer function

$$\mathscr{H}(s) = \mathscr{C}(s\mathscr{E} - \mathscr{A})^{-1}\mathscr{B}.$$

The reduced order model will be obtained from the projection of the original model (1) onto a proper subspace $V_n \in \mathbb{R}^{N\times n}$ with $n \ll N$, i.e.

$$\begin{aligned} V_n^{\mathrm{T}}\mathscr{E}V_n\dot{\tilde{x}}(t) &= V_n^{\mathrm{T}}\mathscr{A}V_n\tilde{x}(t) + V_n^{\mathrm{T}}\mathscr{B}u(t), \\ y(t) &= \mathscr{C}V_n\tilde{x}(t). \end{aligned}$$

The computation of the subspace V_n should result in a small error

$$\|\mathscr{H}(s) - \tilde{\mathscr{H}}(s)\|$$

in terms of a proper norm, where $\tilde{\mathscr{H}}(s)$ denotes the transfer function of the reduced order model.

2 Adaptive Krylov subspace methods

The idea of Krylov subspace methods for model reduction, e.g. [4], results from the expansion of the transfer function

$$\mathscr{H}(s) = \sum_{j=0}^{\infty} Y^{(j)}(s_i)(s - s_i)^j,$$

where $S := \{s_1, \ldots, s_k\}$ denotes a given set of expansion points and $Y^{(j)}(s_i) = \mathscr{C}^{\mathrm{T}}X^{(j)}(s_i)$ with

$$X^{(j)}(s_i) = \left[-(s_i\mathscr{E} - \mathscr{A})^{-1}\mathscr{E}\right]^j (s_i\mathscr{E} - \mathscr{A})^{-1}\mathscr{B}.$$

The orthogonal columns of V_n span the same subspace as

$$\left[X^{(0)}(s_1), \ldots, X^{(j_1)}(s_1), \ldots, X^{(j_k)}(s_k)\right].$$

In [4], the authors present an adaptive choice for the size of the Krylov subspaces $X^{(j)}(s_i)$ applying the rational Arnoldi method. Assuming $Y^{(j)}(s_i) = \hat{Y}^{(j)}(s_i)$ for all $j = 0, \ldots, \hat{J}_{i-1}$ and $s_i \in S$, the Krylov subspace of the expansion point $s_i \in S$ with

$$\max_{s_i\in S} \left| Y^{(\hat{J}_i)}(s_i) - \hat{Y}^{(\hat{J}_i)}(s_i) \right|$$

is increased by one additional vector in each iteration step. Here, $\hat{Y}^{(j)}(s_i)$ denotes the j-th output moment of the reduced order model.
The remaining problem of the adaptive-order rational Arnoldi method (AORA) consists of the adequate choice of the expansion points.

3 AORA with adaptive point selection

We will present a combination of the AORA method and an adaptive expansion point selection. In detail, from the subsequently computed reduced order models using the AORA method new expansion points are determined, until a certain tolerance is reached. The aim of the adaptive expansion point selection consists of the computation of a reduced order model, which offers a good approximation for the whole frequency range.
For the definition of an adequate measurement of the

error $\varepsilon_m = \|\mathscr{H}(s) - \hat{\mathscr{H}}_n(s)\|$, e.g. [3], we define the approximation

$$\hat{\varepsilon}_m = \sum_{l=1}^{m} 2^{l-m} \frac{\|\hat{\mathscr{H}}_l(s) - \hat{\mathscr{H}}_{l-1}(s)\|}{\|\hat{\mathscr{H}}_l(s)\|}, \qquad (2)$$

where $\hat{\mathscr{H}}_{k-1}$ and $\hat{\mathscr{H}}_k$ denote transfer functions of reduced order models obtained from the AORA method. Since this definition does not give a hint, whether $\hat{\varepsilon}_m$ remains small due to convergence or stagnation, we will add one more expansion point in each iteration step.

3.1 Point selection for Maxwell's equations

Due to the high-frequency model problems with the frequency range $\mathscr{I} = [f_{min}, f_{max}]$ the first two expansion points are always defined via $s_1 = i f_{min}$ and $s_2 = i f_{max}$, where i denotes the imaginary unit. Furthermore, all expansion points are purely imaginary. Initially, we usually choose $s_1 = i f_{min}, s_2 = i f_{max}$ and $s_3 = i(f_{min} + f_{max})/2$ as the first set of expansion points S_0.
In the $(k+1)$-th iteration step, the expansion point $s_{k+1} = 2\pi i f_{k+1}$ is determined, such that

$$s_{k+1} = \arg\max_s \frac{\|\hat{\mathscr{H}}_k(s) - \hat{\mathscr{H}}_{k-1}(s)\|}{\|\hat{\mathscr{H}}_k(s)\|}. \qquad (3)$$

Specially, the error during the $(k+1)$-th iteration step (3) is computed alternatingly on the intervals $\mathscr{I}_1 = [f_{min}, (f_{min}+f_{max})/2]$ and $\mathscr{I}_2 = [(f_{min}+f_{max})/2, f_{max}]$. If the error for the given interval $\mathscr{I}_1$ or $\mathscr{I}_2$ is less than a given tolerance $\delta > 0$, we switch back to the other interval and determine a new expansion point. This new expansion point should have a certain distance to previous expansion points from this interval.
Finally, the algorithm terminates, if the global approximation error $\hat{\varepsilon}_k$ reached a given tolerance or the error during the $(k+1)$-th iteration step (3) is less than $\delta > 0$ for both intervals $\mathscr{I}_1$ and $\mathscr{I}_2$.

4 Numerical results

Some numerical results are presented for a coplanar waveguide with a dielectric overlay, where the transmission line is surrounded by two layers of multilayer board. The single input, single output model problem is enclosed in a metallic box and deals with the frequency range $[f_{min}, f_{max}] = [0.6, 3.0]$ GHz.
Here, the discretization of the model problem was carried out using the Finite Integration Technique, e.g. [5], with $N = 32924$ degrees of freedom.
Subsequently reduced order models $\hat{\mathscr{H}}_k(s)$ of dimension $n = 30$ have been computed until $\hat{\varepsilon}_m < 10^{-12}$ applying the tolerance $\delta = 7.0 \cdot 10^{-11}$. The expansion points in the i-th iteration step are denoted by S_i.

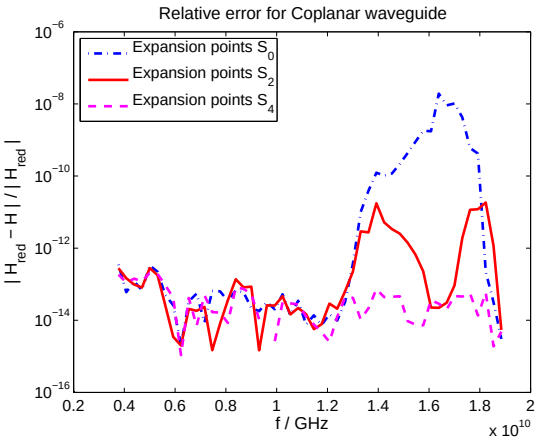

Fig. 1. Relative error for reduced order model with different sets of expansions points.

After four iteration steps the algorithm terminates due to the introduction of $\delta > 0$, where

$$\max_s \|\hat{\mathscr{H}}(s) - \mathscr{H}(s)\| \approx 1.5 \cdot 10^{-11}.$$

Future results will comprise of the application of incomplete multilevel factorizations for the computation of Krylov subspaces, e.g. [1], using previous preconditioning techniques for Helmholtz equations. Furthermore, exisiting adaptive expansion point selections, e.g. for machine tool simulations [2], will be applied to Maxwell's equations. In this context, a combination with the results from section 3 might be adopted.

Acknowledgement. The work is supported by the German Federal Ministry of Education and Research (BMBF), grant no. 03MS613B.

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