Index-aware model Order Reduction: LTI DAEs in electric networks

Nicodemus Banagaaya[1] and Wil Schilders[1]

Dept. of Mathematics and Computer Science, Technische Universiteit Eindhoven, The Netherlands
n.banagaaya@tue.nl, w.h.a.schilders@tue.nl

Summary. Model order reduction (MOR) has been widely used in the electric networks but little has been done to reduce higher index differential algebraic equations(DAEs). Most methods first do an index reduction before reducing a higher DAEs but this can lead to loss of system physical properties. In this paper we present a new MOR method for DAEs called the index-aware MOR (IMOR) which can reduce higher index-2 system while preserving the index of the system.

1 Introduction

Consider a linear time invariant (LTI) DAE system:

$$Ex'(t) = Ax(t) + Bu, \quad x(0) = x_0, \tag{1a}$$

$$y(t) = C^T x(t), \tag{1b}$$

where $E, A \in \mathbb{R}^{n,n}$, $B \in \mathbb{R}^{n,m}$, $C \in \mathbb{R}^{n,\ell}$, $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the input vector, $y(t) \in \mathbb{R}^\ell$ is the output vector and $x_0 \in \mathbb{R}^n$ must be a consistent initial value since E is singular. In many MOR methods [1] they always assume that $x_0 = 0$ which lead to a transfer function $H(s) = C^T(sE - A)^{-1}B$ if and only if matrix pencil $sE - A$ is regular. Unfortunately for the case of DAEs we cannot always have this freedom of choosing an arbitrary initial condition x_0, infact we cannot always obtain a transfer function especially for index greater than 1 as discussed in Sect. 2. This motivated us to propose a new MOR technique for DAEs called the IMOR method which takes care of this limitation [2, 3]. In this technique before we apply MOR we first decompose the DAE system into differential and algebraic parts using matrix and projector chains introduced by März [4] in 1996. We then use the existing MOR techniques such as the Krylov based methods on the differential part and develop new techniques for the algebraic part. This is done as follows: Assume (1a) is of tractability index μ, then it's projector and matrix chains can be written as, set $E_0 := E$, $A_0 := A$, then $E_{j+1} = E_j - A_j Q_j$, $\quad A_{j+1} := A_j P_j$, $\quad j \geq 0$, where $\mathrm{Im} Q_j = \mathrm{Ker} E_j$, $P_j = I_n - Q_j$. There exists μ such that E_μ is nonsingular while all E_j are singular for all $0 \leq j < \mu - 1$. Using these chains we can rewrite Equation (1a) as projected system of index-μ:

$$P_{\mu-1} \cdots P_0 x' + Q_0 x + \cdots + Q_{\mu-1} x = E_\mu^{-1} \left(A_\mu x + Bu \right) \tag{2}$$

In order to decompose higher index systems ($\mu > 1$), März [4] suggested an additional constraint $Q_j Q_i = 0$, $j > i$ on the projector construction. If this constraint holds then Equation (2) can be decomposed into differential and algebraic parts. However, the März decomposition leads to a decoupled system of dimension $(\mu + 1)n$. It does not even preserve the stability the DAE system. This motivated us to modify the März decomposition using special basis vectors as presented in papers [3] and [2] for the case of index-1 and index-2 respectively. Our decomposition leads to a decoupled system of the same dimension as that of the DAE system. Then we apply Krylov methods on the differential part and constructed subspaces to reduce the algebraic parts. In Sect. 2 we briefly discuss the IMOR method for index-2 systems (IMOR-2) more details can be found in [2].

2 Index-aware MOR for index-2 systems

Assume Equation (1a) is an index-2 system this implies $\mu = 2$. We observed that for higher index DAEs there is a possibility of obtaining a purely algebraic decoupled system depending on the nature of spectrum of the matrix pencil $\sigma(E,A) = \sigma_f(E,A) \cup \sigma_\infty(E,A)$, where $\sigma_f(E,A)$ and $\sigma_\infty(E,A)$ is the set of the finite and infinite eigenvalues respectively. This happens when matrix spectrum has only infinite eigenvalues, i.e.$\sigma_f(E,A) = \emptyset$. Thus higher index DAEs can be decomposed into two ways. Due to space we are going to only discuss the case when $\sigma_f(E,A) \neq \emptyset$ the other case can be found in our paper [2]. We now assume matrix pencil of Equation (1a) has atleast one finite eigenvalue. We then construct basis vectors (p,q) in $\mathbb{R}^n$ with their inversion $(p_*, q_*)^T$ for the projectors P_0 and Q_0 respectively where $p \in \mathbb{R}^{n,n_0}$, $q \in \mathbb{R}^{n,k_0}$. This leads to a theorem below.

Theorem 1. *Let $P_{01} = p_*^T P_1 p$, $Q_{01} = p_*^T Q_1 p$, then $P_{01}, Q_{01} \in \mathbb{R}^{n_0,n_0}$ are projectors in $\mathbb{R}^{n_0}$ provided the constraint condition $Q_1 Q_0 = 0$ holds.*

Next, we construct another basis matrix (p_{01}, q_{01}) in $\mathbb{R}^{n_0}$ made of n_{01} independent columns of projector P_{01} and k_1 independent columns of its complementary projector Q_{01} such that $n_0 = n_{01} + k_1$ and it's inverse can be denoted by $(p_{01}^*, q_{01}^*)^T$. Then Equation (1) can be decomposed as:

$$\xi_p' = A_p \xi_p + B_p u, \quad (3a)$$
$$\xi_{q,1} = A_{q,1}\xi_p + B_{q,1}u, \quad (3b)$$
$$\xi_{q,0} = A_{q,0}\xi_p + B_{q,0}u + A_{q,01}\xi_{q,1}', \quad (3c)$$
$$y = C_p^T \xi_p + C_{q,1}^T \xi_{q,1} + C_{q,0}^T \xi_{q,0}, \quad (3d)$$

where

$$A_p := p_{01}^{*T} p_0^{*T} E_2^{-1} A_2 p_0 p_{01}, \quad B_p := p_{01}^{*T} p_0^{*T} E_2^{-1} B,$$
$$A_{q,1} := q_{01}^{*T} p_0^{*T} E_2^{-1} A_2 p_0 p_{01}, \quad B_{q,1} := q_{01}^{*T} p_0^{*T} E_2^{-1} B,$$
$$A_{q,0} := q_0^{*T} P_1 E_2^{-1} A_2 p_0 p_{01}, \quad B_{q,0} := q_0^{*T} P_1 E_2^{-1} B,$$
$$A_{q,01} := q_0^{*T} Q_1 p_0 q_{01}, \quad C_p = p_{01}^T p_0^T C \in \mathbb{R}^{n_{01},\ell},$$
$$C_{q,1} = q_{01}^T p_0^T C \in \mathbb{R}^{k_1,\ell}, \quad C_{q,0} = q_0^T C \in \mathbb{R}^{k_0,\ell}.$$

Equations (3a), (3b) and (3c) are of dimension n_{01},k_1 and k_0 respectively, where $n = n_{01} + k_1 + k_0$. System (3) preserves stability of the DAE system (1) since it can be proved that $\sigma(A_p) = \sigma_f(E,A)$. If we take the Laplace transform of (3) and set $\xi_p(0) = 0$ then we obtain

$$Y(s) = \left[H_p(s) + H_{q,1}(s) + H_{q,0}(s)\right] U(s) + H_{q,0}(0),$$

where $H_p(s) = C_p^T (sI_{n_0} - A_p)^{-1} B_p$,
$H_{q,1}(s) = C_{q,1}^T \left[A_{q,1}(sI_{n_0} - A_p)^{-1} B_p + B_{q,1}\right]$,
$H_{q,0}(s) = C_{q,0}^T \left[(A_{q,0} + sA_{q,01}A_{q,1})(sI_{n_0} - A_p)^{-1} B_p\right]$
$+ C_{q,0}^T \left[B_{q,0} + sA_{q,01}B_{q,1}\right], H_{q,0}(0) = -C_{q,0}^T A_{q,01} B_{q,1} u(0)$.
Thus not always we can obtain the transfer function of index 2 systems for arbitrary input vector u unless $H_{q,0}(0) = 0 \Rightarrow Y(s) = H(s)U(s)$. We can now apply IMOR-2 method as follows: If we choose the expansion point $s_0 \in \mathbb{C} \setminus \sigma(A_p)$, we construct a Krylov-subspace generated by $M_p := -(s_0 I_{n_0} - A_p)^{-1}$ and $R_p := (s_0 I_{n_0} - A_p)^{-1} B_p$. Then, $V_{p_r} := orth(\kappa_r(M_p, R_p))$, $r \le n_{01}$. We then use V_{p_r} to construct the subspace $\mathscr{V}_{q,1} = \mathrm{span}(B_{q,1}, A_{q,1}V_{p_r})$ and its orthonormal matrix is denoted by $V_{q_{\tau_1},1} = orth(\mathscr{V}_{q,1})$, $\tau_1 \le \min((r+1)m, \dim(\mathscr{V}_{q_{\tau_1}}))$. We finally construct subspace
$\mathscr{V}_{q,0} = \mathrm{Span}\{\mathscr{V}_{Q_1}, \mathscr{V}_{Q_2}, \mathscr{V}_{Q_3}\}$, where
$\mathscr{V}_{Q_1} = A_{q,0}R_p + B_{q,0} + s_0(A_{q,01}A_{q,1}R_p + A_{q,01}B_{q,1})$,
$\mathscr{V}_{Q_2} = A_{q,01}B_{q,1}$,
$\mathscr{V}_{Q_3} = \left[(A_{q,0} + s_0 A_{q,01}A_{q,1})M_p + A_{q,01}A_{q,1}\right] V_{p_r}$ and it's orthonormal matrix is denoted by $V_{q_{\tau_0},0} = orth(\mathscr{V}_{q,0})$, where $\tau_0 \le \min((r+2)m, \dim(\mathscr{V}_{q,0}))$. We can now use the orthonormal matrices V_{p_r},$V_{q_{\tau_1},1}$ and $V_{q_{\tau_0},0}$ to reduce the dimension of the subsystems (3a), (3b) and (3c) respectively as consequence the dimension of the decoupled system (3) is also reduced. Hence, if we substitute $\xi_p = V_{p_r}\xi_{p_r}$, $\xi_{q,1} = V_{q_{\tau_1},1}\xi_{q_{\tau_1},1}$,
$\xi_{q,0} = V_{q_{\tau_0},0}\xi_{q_{\tau_0},0}$, into system (3) and simplifying we can obtain a reduced model of DAE system (1) which will call the IMOR-2 model.

3 Numerical results

We used an index -2 test system called S8OPI in [5] which is a large power system RLC model. It's a single-input single-output (SISO) system of dimension 4182. We applied the IMOR-2 method using $s_0 = j10^3$. We obtained a reduced model of total dimension 219 as shown in Table 1. We observed that the magnitude of the transfer reduced model coincides with that of the original model at low frequencies with very small error as shown in Fig. 1. We have seen that the IMOR-2 method leads to good reduced model and can be used on any index-2 system.

Table 1. Dimension of the Original and Reduced model

Models	Dimension		
	n_{01}	k_1	k_0
Original Model	4028	35	119
Reduced Model	170	1	48

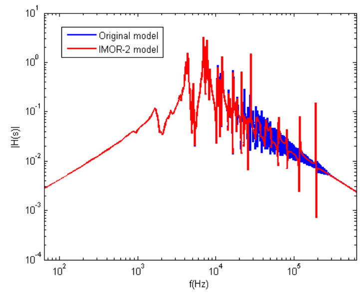

Fig. 1. Magnitude of the transfer functions

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