Automatic model order reduction by moment-matching according to an efficient output error bound

Lihong Feng and Peter Benner

Max Planck Institute for Dynamics of Complex Technical Systems, Germany feng@mpi-magdeburg.mpg.de, benner@mpi-magdeburg.mpg.de

Summary. An output error bound is proposed for model order reduction of linear time invariant (LTI) systems. According to the error bound, the model order reduction method based on moment-matching (moment-matching MOR) can be implemented by selecting the expansion points adaptively, such that the reduced model can be obtained automatically. The error bound is an estimation for the error between the transfer function of the original system and that of the reduced model. Simulation results show the efficiency of the error bound.

1 Introduction

Consider an LTI system

$$\begin{aligned} E\tfrac{d\mathbf{x}}{dt} &= A\mathbf{x}+Bu(t), \\ y(t) &= C\mathbf{x}. \end{aligned} \tag{1}$$

If we use moment-matching MOR, usually we apply the Laplace transform to (1), and get

$$\begin{aligned} &sE\mathbf{x}(s)-A\mathbf{x}(s)=Bu(s), \\ &y(s)=C\mathbf{x}(s). \end{aligned} \tag{2}$$

From the series expansion of $\mathbf{x}(s)$,

$$\mathbf{x}(s)=\sum_{i=0}^{\infty}[-(s_0E-A)^{-1}E]^i(s_0E-A)^{-1}BU(s)(s-s_0)^i, \tag{3}$$

the matrix V is computed as

$$\begin{aligned} \text{range}\{V\} = \text{span}\{&\tilde{B}(s_0),\ \tilde{A}(s_0)\tilde{B}(s_0), \\ &\ldots,\ (\tilde{A}(s_0))^q\tilde{B}(s_0)\}, \end{aligned} \tag{4}$$

where $\tilde{A}(s_0)=(s_0E-A)^{-1}E$, $\tilde{B}(s_0)=(s_0E-A)^{-1}B$ and $q \ll n$. The reduced model is

$$\begin{aligned} V^TEV\tfrac{d\mathbf{z}}{dt} &= V^TAV\mathbf{z}+V^TBu(t), \\ y(t) &= CV\mathbf{z}. \end{aligned} \tag{5}$$

Instead of using single-point expansion s_0, one can use multi-point expansion to compute V. That is, choosing multiple expansion points $s_i,\ i=0,1,\ldots m$, we compute each matrix V_i corresponding to s_i according to (4). Finally, $V=\text{orthogonalize}\{V_1,\ldots,V_m\}$.

By using multi-point expansion, the error of the reduced model can be kept small in a wider frequency range. At present, how to adaptively choose the expansion points s_i is under investigation using several points of view. We aim to derive an error bound for the transfer function $\hat{H}(s)$ of the reduced model, such that the expansion points can be adaptively chosen according to the error bound. Since the transfer function can be considered as the impulse response of the LTI system in frequency domain, the error bound can be considered as the output error bound in frequency domain.

The error bound is motivated by the idea in [1], where an output error bound for the weak form of a parametrized Partial Differential Equation (PDE) is derived. The error bound in [1] is obtained in the function space for the weak form, where all the parameters in the PDE must be real variables. Since moment-matching MOR directly deal with the discretized system (2) in the vector space, it is best that an error bound is derived in the vector space rather than in the function space. Moreover, system (2) can be seen as a parametrized system with parameter s being a complex variable.

In summary, in order to obtain the error bound for $\hat{H}(s)$, the method in [1] is not valid due to the challenges below:

1. The error bound should be derived in the vector space C^n.
2. The error bound should be valid for complex parameters.

Method for deriving the output error bound must be adapted in order to meet the above two challenges.

2 Output Error Bound for an LTI System

We first present the analysis for single-input single-output (SISO) systems, then extend the result to multiple-input multiple-output (MIMO) systems.

We assume that the matrix $G(s)=sE-A$ satisfies

$$Re(\mathbf{x}^*G(s)\mathbf{x}) \geq \alpha(s)(\mathbf{x}^*\tilde{A}\mathbf{x}), \tag{6}$$

and

$$Im(\mathbf{x}^*G(s)\mathbf{x}) \geq \gamma(s)(\mathbf{x}^*\tilde{A}\mathbf{x}), \tag{7}$$

where $Re(\cdot)$ means the real part of $\mathbf{x}^*G(s)\mathbf{x}$, and $Im(\cdots)$ is the imaginary part. $\alpha(s),\gamma(s)$: $C \to R_+$ may depend on the parameter s. The matrix $\tilde{A}=s_0E-A$ is assumed to be symmetric, positive definite, which is satisfied by many engineering problems.

For systems with $C \neq B^T$, we need to define a dual system in frequency domain,

$$\begin{aligned} \bar{s}E^*\mathbf{x}^{du}(s) - A^*\mathbf{x}^{du}(s) &= -C^*, \\ y^{du} &= B^*\mathbf{x}^{du}(s). \end{aligned} \quad (8)$$

Let $r^{pr}(s) = B - G(s)\hat{\mathbf{x}}(s)$ be the residual for the primal system in (2), and $r^{du}(s) = -C^* - (\bar{s}E^* - A^*)\hat{\mathbf{x}}^{du}(s) = G^*(s)\hat{\mathbf{x}}^{du}(s)$ be the residual for the dual system. We will show that $r^{pr}(s)$ can be represented through a vector $\hat{\varepsilon}^{pr} \in \mathbb{C}^n$, and $r^{du}(s)$ can be represented through a vector $\hat{\varepsilon}^{du} \in \mathbb{C}^n$.

Define a function $f^{pr}(\xi) = \xi^* r^{pr}(s) : \mathbb{C}^n \to \mathbb{C}$ for the primal system. From the Riez representation theorem, there exists a unique vector $\hat{\varepsilon}^{pr} \in \mathbb{C}^n$, such that

$$f^{pr}(\xi) = \langle \hat{\varepsilon}^{pr}, \xi \rangle = \xi^* \tilde{A} \hat{\varepsilon}^{pr}. \quad (9)$$

We also define a function $f^{du}(\xi) = (r^{du}(s))^* \xi \colon \mathbb{C}^n \to \mathbb{C}$. Similarly, there exists a unique vector $\hat{\varepsilon}^{du} \in \mathbb{C}^n$, such that

$$f^{du}(\xi) = \langle \hat{\varepsilon}^{du}, \xi \rangle = \xi^* \tilde{A} \hat{\varepsilon}^{du}. \quad (10)$$

Theorem 1. *If the reduced model of the primal system (2) and that of the dual system (8) is computed by the same projection matrix V, the matrices E, A are symmetric, and $G(s)$ satisfies (6), and (7), then $-S_R - \beta_R \leq Re(H(s) - \hat{H}(s)) \leq S_R - \beta_R$ and $-S_I - \beta_I \leq Im(H(s) - \hat{H}(s)) \leq S_I - \beta_I$. Here,*
$\beta_R = \frac{1}{4\alpha(s)}(\hat{\varepsilon}^{pr})^*\tilde{A}\hat{\varepsilon}^{du} + \frac{1}{4\alpha(s)}(\hat{\varepsilon}^{du})^*\tilde{A}\hat{\varepsilon}^{pr}$, $\beta_I = \frac{\alpha(s)}{\gamma(s)}\beta_R$,
$S_R = \frac{1}{2\alpha(s)}\sqrt{(\hat{\varepsilon}^{pr})^*\tilde{A}\hat{\varepsilon}^{pr}}\sqrt{(\hat{\varepsilon}^{du})^*\tilde{A}\hat{\varepsilon}^{du}}$, $S_I = \frac{\alpha(s)}{\gamma(s)}S_R$.

From Theorem 1, we get the error bound,

$$|H(s) - \hat{H}(s)| \leq \sqrt{B_R^2 + B_I^2} := \Delta(s), \quad (11)$$

where $B_R = max\{|S_R - \beta_R|, |S_R + \beta_R|\}$, $B_I = max\{|S_I - \beta_I|, |S_I + \beta_I|\}$. The error bound $\Delta(s)$ can be computed cheaply though it is dependent on the parameter s, because the main computational part for $\Delta(s)$ is independent of s, which can be implemented off-line. If relative error is preferred, one should use $\Delta_{re}(s) = \Delta(s)/\hat{H}(s)$. For MIMO systems, assume $H_{ij}(s)$ is the transfer function corresponding to the ith input and jth output. For each pair of i, j, we can compute $\Delta_{ij}(s)$. The error bound $\Delta(s)$ can be defined as $\Delta(s) = \max_{ij} \Delta_{ij}(s)$.

3 Adaptively Choosing Expansion Points

From the construction of the error estimator $\Delta(s)$, the projection matrix V can be constructed by the algorithm as below,

Algorithm 1 *$V = []$;*
Choose initial s^*;
$\varepsilon = 1$;
While $\varepsilon \geq \varepsilon_{tol}$ ($\varepsilon_{tol} < 1$ is the error tolerance.)
$range(V) = range(V) + span\{\tilde{B}(s^*), \tilde{A}(s^*)\tilde{B}, \ldots, \tilde{A}^q(s^*)\tilde{B}\}$;
$s^* = arg \max_{s \in \Xi_{train}} \Delta(s)$; ($\Xi_{train}$ is the sample space for s.) ;
$\varepsilon = \Delta(s^*)$;
End While

4 Simulation Results

We take two examples to support the theoretical analysis above. One example is a spiral inductor, a SISO system; the other is an optical filter, a system with 5 outputs. Both examples are taken from the Oberwolfach Benchmark Collection (URL: http://simulation.uni-freiburg.de/downloads/benchmark).

Define $\varepsilon_{max} = \max_{ij} \max_{k} |H_{ij}(s_k) - \hat{H}_{ij}(s_k)|/|\hat{H}_{ij}(s_k)|$ as the maximal true error of the current $\hat{H}(s)$ over 2000 sample points, and it is used as the error of the current reduced model. Results of Algorithm 1 for the spiral inductor is listed in Table 1. r is the order of the reduced model. After 4 iterations, four expansion points have been selected, a reduced model with accuracy $O(10^{-8})$ is obtained. Figure 1 plots ε_{max} vs. the error bound $\Delta_{re}(s)$ for the multi-output system, showing $\Delta_{re}(s)$ performs well, especially at the latter iterations.

Table 1. Spiral inductor $q = 5$, $\varepsilon_{tol} = 10^{-3}$, $n = 1434$, $r = 24$

iteration	$s^*/(j\omega)$	ε_{max}	$\Delta_{re}(s^*)$
1	1×10^{10}	0.30	86.99
2	3.43×10^7	0.04	16.15
3	3.39×10^8	7×10^{-5}	6×10^{-3}
4	1.41×10^9	7.73×10^{-8}	7.50×10^{-6}

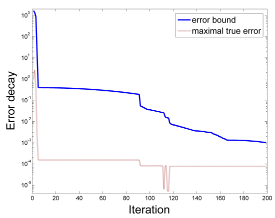

Fig. 1. Optical filter, $q = 1$, $\varepsilon_{tol} = 10^{-3}$, $n = 1668$, $r = 21$.

References

1. D. V. Rovas. *Reduced-Basis Output Bound Methods for Parametrized Partial Differential Equations*. PhD thesis, Massachusetts Institute of Technology, 2003.