Robust transmission conditions of high order for thin conducting sheets

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Summary. Resolving thin conducting sheets for shielding or even skin layers inside by the mesh of numerical methods like the finite element method (FEM) can be avoided by using impedance transmission conditions (ITCs). Those ITCs shall provide an accurate approximation for small sheet thicknesses d, where the accuracy is best possible independent of the conductivity or the frequency being small or large – this we will call *robustness*. We investigate the accuracy and robustness of popular [1, 2] and recently developed ITCs [4], and propose robust ITCs which are accurate up to $O(d^2)$.

1 Introduction

Thin conducting sheets for the protection of electronic devices exhibit large ratios of characteristic lengths which require a small mesh size when using finite difference or finite element schemes. Besides this issue of computational cost due to the small geometry detail, many commercial mesh generators get difficulties with anisotropic geometrical features.

The shielding behaviour can be modelled alternatively by replacing the thin sheet by an interface on which *impedance transmission conditions* are set.

We consider the time-harmonic eddy current model (convention $\exp(-i\omega t)$, $\omega > 0$) in two dimensions

$$\mathbf{curl}_{2D}\, e(\mathbf{x}) = i\omega\mu_0 \mathbf{h}(\mathbf{x}), \qquad (1)$$

$$\mathrm{curl}_{2D}\, \mathbf{h}(\mathbf{x}) = \sigma e(\mathbf{x}) + j_0(\mathbf{x}) \qquad (2)$$

where e and $\mathbf{h}$ are the out-of-plane electric and in-plane magnetic fields, σ is the conductivity of the thin sheet of thickness d and zero elsewhere, and j_0 is the out-of-plane imposed current which is outside the conductor. We have used the 2D rotation operators $\mathbf{curl}_{2D} = (\partial_y, -\partial_x)^\top$ and $\mathrm{curl}_{2D} = (-\partial_y, \partial_x)$. The skin depth inside the conductor is $\delta = \sqrt{2/\omega\mu_0\sigma}$.

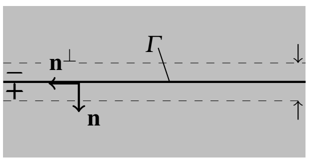


Fig. 1. Impedance transmission conditions are set on the mid-line Γ of the sheet and shall approximate the exact field outside the area the sheet was originally located.

2 Thin sheet and limit conditions

2.1 Thin sheet transmission conditions

With $\beta = i\omega\mu_0$ and $\gamma = \sqrt{-i\omega\mu_0\sigma}$ the impedance transmission conditions by Krähenbühl and Muller [1] and Mayergoyz and Bedrosian [2] are given by

$$\begin{aligned} e^+_{\mathrm{KM}} - e^-_{\mathrm{KM}} &= \tfrac{\beta}{\gamma}\tanh(\tfrac{\gamma d}{2})(\mathbf{h}^+_{\mathrm{KM}}\cdot\mathbf{n} + \mathbf{h}^-_{\mathrm{KM}}\cdot\mathbf{n}),\\ \mathbf{h}^+_{\mathrm{KM}}\cdot\mathbf{n}^\perp - \mathbf{h}^-_{\mathrm{KM}}\cdot\mathbf{n}^\perp &= \tfrac{\gamma}{\beta}\tanh(\tfrac{\gamma d}{2})(e^+_{\mathrm{KM}} + e^-_{\mathrm{KM}}) \end{aligned} \qquad (3)$$

which are set on the mid-line Γ of the thin sheet. Here, the subscript KM denotes the approximative electric and magnetic field, the superscript $\pm$ denotes the values on the two sides of the sheet, and $\mathbf{n} = (n_1, n_2)^\top$ and $\mathbf{n}^\perp = (n_2, -n_1)^\top$ are the normalised normal and tangential vectors on Γ like shown in Fig. 1.

2.2 The limit of vanishing thickness

Impedance transmission conditions are developed for thin sheets and their accuracy shall be larger the thinner the sheet. We observe three different limits for vanishing sheet thickness ($d \to 0$):

1. The conductivity σ is remained or is increased less than $1/d$. Then, we have twofold continuity

$$\begin{aligned} e^+_0 - e^-_0 &= 0,\\ \mathbf{h}^+_0\cdot\mathbf{n}^\perp - \mathbf{h}^-_0\cdot\mathbf{n}^\perp &= 0. \end{aligned} \qquad (4)$$

   The limit corresponds to the low-frequency eddy current limit $\delta \to \infty$.
2. The conductivity σ increases like $1/d$, where we get the non-trivial limit conditions [3]

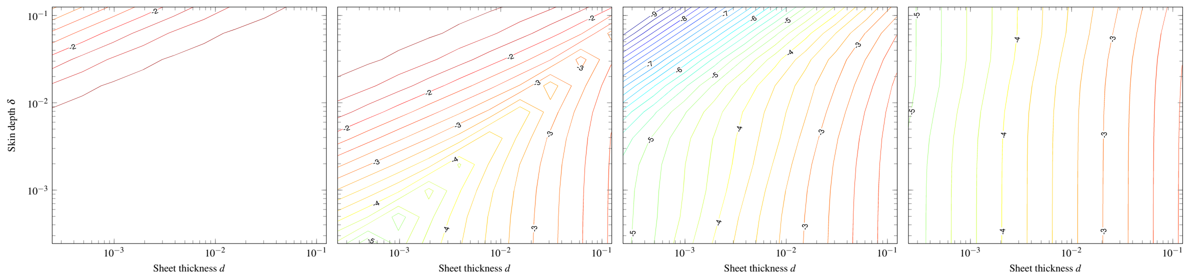
$$\begin{aligned} e^+_1 - e^-_1 &= 0,\\ \mathbf{h}^+_1\cdot\mathbf{n}^\perp - \mathbf{h}^-_1\cdot\mathbf{n}^\perp &= \tfrac{\sigma d}{2}(e^+_1 + e^-_1). \end{aligned} \qquad (5)$$

3. The conductivity σ increases more than $1/d$, *e. g.*, like $1/d^2$. Then, the electric field on both sides get zero in the limit $d \to 0$,

$$e^+_2 = e^-_2 = 0, \qquad (6)$$

   equivalently to the high-frequency limit $\delta \to 0$.

Here, the respective subscripts 0, 1 and 2 correspond to the scaling $\sigma \sim 1/d^\alpha$ with $\alpha = 0, 1, 2$.



(a) Low-frequency limit. (b) High-frequency limit. (c) Non-trivial limit. (d) Thin sheet conditions.

Fig. 2. The modelling error when using (a) the low-frequency limit, (b) the high-frequency limit, (c) the non-trivial limit, and (d) the thin sheet conditions (3) to approximate the shielding of conducting sheets of different thicknesses d and different frequencies or skin depth δ. The shown error is the one of the magnetic field outside the thin sheet.

2.3 Discussion

We investigated the thin sheet conditions and the limit conditions with high-order finite elements for a straight thin sheet in a rectangular box (with periodic boundary conditions) and two circular live wires with opposite current direction. The original thin sheet conditions (3) turn out to be robust with respect to the skin depth or frequency, see Fig. 2(d), which is obvious as they transform into (4) for low frequencies and into (6) for high frequencies, cf. [1].

The low-frequency limit conditions (4) achieve only some accuracy if the sheet thickness is more than one or two orders smaller than the skin depth. The high-frequency limit conditions (6) entail some accuracy if the skin depth is at least at the order of the sheet thickness.

The non-trivial limit conditions (5) are again robust and their accuracy is comparable to the one of the original thin sheet conditions for large skin depths / thickness ratios and much better if the skin depth gets relatively small, see Fig. 2(c). This observation is remarkable as the expression of (5) is much simpler than the one of (3).

3 High order transmission conditions

In order to improve the accuracy we have studied an asymptotic expansion for $d \to 0$ where – motivated by the non-trivial limit conditions – the conductivity is once scaled like $1/d$ (case $\alpha = 1$) and – motivated by asymptotically constant skin depth – the conductivity is once scaled like $1/d^2$ (case $\alpha = 2$).

3.1 Conductivity scaled like $1/d$

The first order ITCs related to $\alpha = 1$ are given by [4]

$$\begin{aligned} e_{1,1}^{+} - e_{1,1}^{-} &= 0, \\ \mathbf{h}_{1,1}^{+} \cdot \mathbf{n}^{\perp} - \mathbf{h}_{1,1}^{-} \cdot \mathbf{n}^{\perp} &= \tfrac{\sigma d}{2}\left(1 + \tfrac{1}{6} i\omega\mu_0\sigma d^2\right)\left(e_{1,1}^{+} + e_{1,1}^{-}\right). \end{aligned}$$

The second and third ITCs involve curvature terms and second order tangential derivatives, see [4].

3.2 Conductivity scaled like $1/d^2$

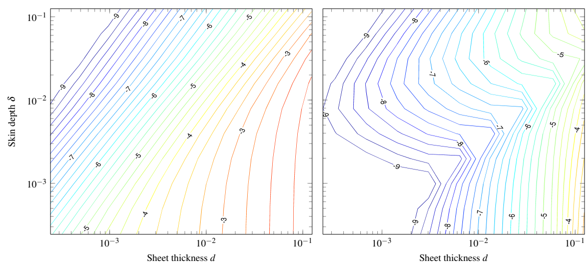
The first order ITCs related to $\alpha = 2$ are given by

$$\begin{aligned} e_{2,1}^{+} - e_{2,1}^{-} &= \frac{\beta d}{2}\left(1 - \frac{\tanh(\frac{\gamma d}{2})}{\frac{\gamma d}{2}}\right)\left(\mathbf{h}_{2,1}^{+} \cdot \mathbf{n} + \mathbf{h}_{2,1}^{-} \cdot \mathbf{n}\right), \\ \mathbf{h}_{2,1}^{+} \cdot \mathbf{n}^{\perp} - \mathbf{h}_{2,1}^{-} \cdot \mathbf{n}^{\perp} &= \frac{\gamma}{\beta} \frac{\sinh(\frac{\gamma d}{2})}{\cosh(\frac{\gamma d}{2}) - \frac{\gamma d}{2}\sinh(\frac{\gamma d}{2})}\left(e_{2,1}^{+} + e_{2,1}^{-}\right). \end{aligned}$$

Additional terms will be present for curved sheets.

3.3 Discussion

Both proposed ITCs are robust and get improved accuracy in comparison to the non-trivial limit and the original thin sheet conditions. The accuracy for both ITCs is asymptotically like $O(d^2)$. Especially, the $\alpha = 2$-ITCs achieve accurate results even for larger sheet thicknesses. Since their expression has the same form as the original thin sheet conditions (3) they are preferable – for low and for high frequencies.



(a) Order 1 $\alpha = 1$. (b) Order 1 $\alpha = 2$.

Fig. 3. Error of the impedance models of order 1 derived by asymptotic expansion for the scaling $\sigma \sim 1/d^{\alpha}$, $\alpha = 0, 1$.

References

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