Efficient Convolution Based Impedance Boundary Conditions

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Summary. When formulating impedance boundary conditions in time domain, the Dirichlet-to-Neumann map of the interior of a good conductor involves convolutions. A. Schädle, M. López-Fernández and C. Lubich have developed a fast and memory efficient algorithm based on Runge-Kutta methods for computing convolutions when only the Laplace transform of the kernel is known. We investigate the coupling of FCQ with FEM for solving parabolic PDE with impedance boundary conditions involving convolutions.

1 Introduction

Alternating electromagnetic fields decay exponentially when penetrating a good conductor (skin effect). Therefore, a reasonable approximation of the electromagnetic Dirichlet-to-Neumann map of the interior of a good conductor is provided by the impedance boundary conditions

$$\big(\mathbf{curl}\mathcal{L}(\mathbf{E})(s)\big) \times \mathbf{n} = \frac{\sqrt{2\mu\sigma s}}{(1-\mathrm{i})\sqrt{\mathrm{i}}}\, \gamma_{\mathrm{D}}\mathcal{L}(\mathbf{E})(s), \qquad (1)$$

where $\mathcal{L}(\mathbf{E})(s)$ denotes the temporal Laplace transform of the electric field, s is a complex variable and γ_{D} is the tangential Dirichlet trace operator. The conductivity σ and permeability μ are known material parameters.

The relationship (1) is valid in the Laplace domain only. When formulating impedance boundary conditions in the time domain, we encounter temporal convolutions of the form

$$\mathbf{curl}\ \mathbf{E}(\mathbf{x},t) \times \mathbf{n} = \int_{t_0}^{t} k(\mu,\sigma,\tau - t)\gamma_{\mathrm{D}}\mathbf{E}(\mathbf{x},\tau)d\tau. \quad (2)$$

2 Fast Convolution Quadrature

C. Lubich and A. Ostermann first introduced the Runge-Kutta convolution quadrature in [1]. Their algorithm requires only the knowledge of the Laplace transform K of the possibly weakly singular kernel and experiences excellent stability properties and high order of convergence.

Subsequently in [2] A. Schädle, M. López-Fernández and C. Lubich rearranged the computations and combined the convolution quadrature with the exponentially convergent approximation of the convolution weights along hyperbolae. They obtained a fast and memory efficient algorithm which virtually shares the convergence and stability properties of the convolution quadrature. Table 1 compares the complexity of a naive implementation of the convolution quadrature with the reduced complexity of the FCQ.

Table 1. Complexity of Convolution Quadrature and FCQ, n indicates the number of timesteps.

	CQ	FCQ
multiplications	$\mathcal{O}(n^2)$	$\mathcal{O}(n\log n)$
evaluations of K	$\mathcal{O}(n)$	$\mathcal{O}(\log n)$
active memory	$\mathcal{O}(n)$	$\mathcal{O}(\log n)$

3 FEM-FCQ Coupling

We have investigated the coupling of the FEM and the FCQ for solving the exterior eddy current problem. The algorithm benefits from the computational efficiency of the FCQ and seems to inherit the good convergence and stability properties which both the FEM and the FCQ supply.

For example we have combined the linear Lagrangian FEM on a triangular mesh with nodal basis functions with the RadauIIA based FCQ for solving the eddy current problem, after assuming a translation symmetry of the model and the TE-mode. We have observed that both the maximal order of convergence in space of FEM an the maximal order of convergence in time of FCQ have been achieved.

References

1. Ch. Lubich and A. Ostermann. Runge-Kutta methods for parabolic equations and convolution quadrature. *Math. Comp.*, 60(201):105–131, 1993.
2. Achim Schädle, María López-Fernández, and Christian Lubich. Fast and oblivious convolution quadrature. *SIAM J. Sci. Comput.*, 28(2):421–438 (electronic), 2006.