The (1+1)D Space-Time Discontinuous Galerkin Trefftz Method

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Summary. A novel Discontinuous Galerkin Finite Element Method for space–time electrodynamic problems is presented. The method employs space–time Trefftz basis functions that satisfy the underlying partial differential equations exactly in an element–wise fashion. A major advantage of Trefftz approximations is their high accuracy that in many cases leads to spectral convergence. First computational results are presented.

1 Introduction

Discontinuous Galerkin Finite Element Methods (DG-FEM) [1–4] are a major class of tools to numerically simulate complicated Electro–Magnetic (EM) systems. Here we present a highly accurate type of DG-FEM. A distinguishing new feature of the method is the use of Trefftz basis functions instead of the traditional generic polynomials. By definition space-time Trefftz basis functions satisfy Maxwell's equations exactly in an element-wise fashion. The method is, hence, a Discontinuous Galerkin Trefftz Finite Element Method (DGT-FEM) [5]

2 Development of the Method

This section consists of three parts. First, we state Maxwell's equations in (1+1)D. Second, we derive a weak formulation of Maxwell's equations and finally introduce Trefftz-type basis functions.

2.1 Maxwell's Equations in 1D

For a wave traveling in the x-direction, with electric and magnetic fields polarized as $E := E_y$ and $H := H_z$, we can write Maxwell's equations in the one dimensional form

$$\begin{pmatrix}\partial_t\\ \partial_x\end{pmatrix}^{\mathbf{T}} \cdot \begin{pmatrix}\varepsilon & 0\\ 0 & 1\end{pmatrix} \cdot \begin{pmatrix}E\\ H\end{pmatrix} = 0,$$
$$\begin{pmatrix}\partial_t\\ \partial_x\end{pmatrix}^{\mathbf{T}} \cdot \begin{pmatrix}0 & \mu\\ 1 & 0\end{pmatrix} \cdot \begin{pmatrix}E\\ H\end{pmatrix} = 0. \tag{1}$$

Here μ is the magnetic permeability and ε is the dielectric permittivity. We assume the space-time domain of interest Ω to be free of any sources. With the abbreviations

$$\eta_\varepsilon := \begin{pmatrix}\varepsilon & 0\\ 0 & 1\end{pmatrix},\ \eta_\mu := \begin{pmatrix}0 & \mu\\ 1 & 0\end{pmatrix} \text{ and } \nabla := \begin{pmatrix}\partial_t\\ \partial_x\end{pmatrix},$$

we cast Maxwell's equations into the form

$$\nabla^{\mathbf{T}} \cdot \eta_\varepsilon \cdot \mathbf{F} = 0 \quad \text{and} \quad \nabla^{\mathbf{T}} \cdot \eta_\mu \cdot \mathbf{F} = 0. \tag{2}$$

Here the EM field vector $\mathbf{F}$ reads

$$\mathbf{F} := \begin{pmatrix}E\\ H\end{pmatrix}.$$

2.2 Weak DG-Form of Maxwell's Equations

We obtain the weak form of (2) by multiplying (2) with a vectorial test function

$$\mathbf{v} := \begin{pmatrix}v^E\\ v^H\end{pmatrix}.$$

and integrating over the domain of interest. This leads to the following form

$$\int_\Omega \left(\nabla^{\mathbf{T}} \cdot \eta_\varepsilon \cdot \mathbf{F}\right) v^H dA + \int_\Omega \left(\nabla^{\mathbf{T}} \cdot \eta_\mu \cdot \mathbf{F}\right) v^E dA = 0,$$

After integration by parts and a subsequent application of the Gauss Theorem the weak form of Maxwell's equations reads

$$\int_{\partial\Omega} v^H \left(\eta_\varepsilon \cdot \mathbf{F}\right) \cdot \mathbf{n} d\Gamma - \int_\Omega \left(\nabla^{\mathbf{T}} v^H\right) \cdot \eta_\varepsilon \cdot \mathbf{F} dA$$
$$+ \int_{\partial\Omega} v^E \left(\eta_\mu \cdot \mathbf{F}\right) \cdot \mathbf{n} d\Gamma - \int_\Omega \left(\nabla^{\mathbf{T}} v^E\right) \cdot \eta_\mu \cdot \mathbf{F} dA = 0. \tag{3}$$

where $\mathbf{n}$ is the unit normal on the space-time domain boundary $\Gamma := \partial\Omega$.

2.3 The Trefftz Basis

Standard FEM uses generic polynomials as basis functions. However, problem–specific basis functions, especially Trefftz-type functions [6] can provide much better accuracy . Here, we use vectorial basis functions whose components are transport polynomials (see Fig. 1) of the form

$$\mathbf{u}^{p,\pm} = \begin{pmatrix} u^{E,p,\pm} = \pm\left(x \pm vt\right)^p \\ u^{H,p,\pm} = Z\left(x \pm vt\right)^p \end{pmatrix}, \tag{4}$$

where $Z = \sqrt{\frac{\mu}{\varepsilon}}$ is the intrinsic impedance and v the speed of light in the medium. The basis function $\mathbf{u}^{p,+}$ corresponds to a wave traveling leftward whereas $\mathbf{u}^{p,-}$ corresponds to a wave traveling rightward; p is the order of the basis function. The field vector is a linear combination of the Trefftz waves

$$\mathbf{F} = \sum_{p=0}^{P} f^p \left(\mathbf{u}^{p,+} + \mathbf{u}^{p,-}\right), \tag{5}$$

where P is the maximum order of approximation and f^p is the field coefficient of order p. Therefore the total number of coefficients f^p is $2(1+P)$, each corresponding to a vectorial basis function $\mathbf{u}^p$.

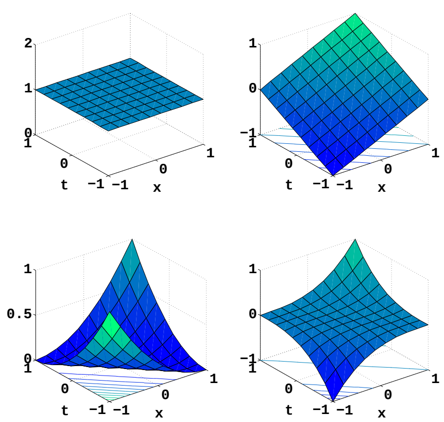

Fig. 1. The first four transport polynomials of order $p = 0, p = 1, p = 2$ and $p = 3$ plotted in a computational space-time domain $(x,t) \in [-1,1] \times [-1,1]$

3 Results

As a first test of the new method, we simulate a Gaussian wave in a domain with an interface between two media at $x = -5$. For obtaining Fig. 2 we set $P = 10$, $N_s = 30$ and $N_t = 60$. The medium left of $x = -5$ is a medium with $\mu = 1$ and $\varepsilon_r = 16$. The space–time solution shows the right behavior in each medium. At the interface a partial reflexion occurs with the right amplitudes of the reflected and transmitted waves. Also the speed-of-light in the medium changes (by a factor of four) resulting in a different trace-angles. In Fig. 3 the relative error of the vacuum simulation is plotted against the number of the Degrees of Freedom (DoF). We obtain exponential convergence of the relative error measured in the $\mathscr{L}^2$ norm.

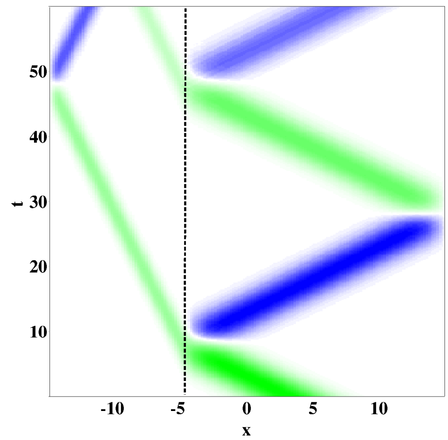

Fig. 2. The electric field of a 1D Gaussian wave, simulated with the DGT-FEM. The solution in the whole space-time domain of interest $(x,t) \in [-15,15] \times [0,60]$ is displayed. A medium interface is set at $x = -5$.

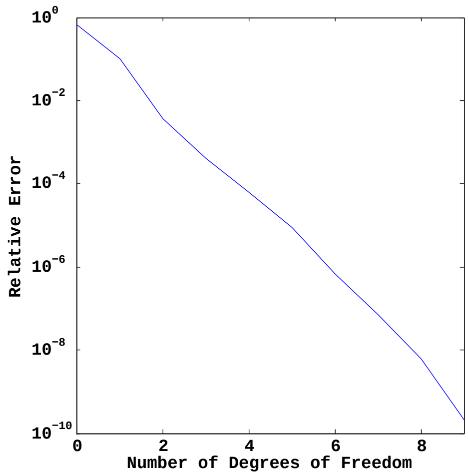

Fig. 3. The relative error of the vacuum simulation plotted against the number of the Degrees of Freedom.

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