An Embarrassingly Parallel Algorithm for Finite Element Formulations

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Summary. In this paper we derive a new stepwise embarrassingly parallel, globally convergent algorithm for linear and non-linear electrostatic problems. It is based on a new interpretation of the classical finite element formulation. We show problems with cellwise linear materials and prove convergence of the proposed method.

1 Overview

Nodal search finite element methods have been derived in [1] using a non-Galerkin approach. It allows a natural decoupling of finite elements similarly to discontinuous Galerkin approaches [2]. This paper will lay out this algorithm using a variation of the usual variational Galerkin method based finite element approach (see e.g. [3]) for equation 1.

$$div(\varepsilon grad(\phi)) = \rho \tag{1}$$

Using the Galerkin finite element approach (see e.g. [3]) has the advantage that it is easier to apply, broadly known, and seems to yield better conditioned systems, although it is less general then the initial approach in [1].

1.1 A Special Finite Element Representation

The proposed algorithm is based on a Lagrange finite element formulation, in which the resulting problem, after using the Galerkin approach, has the form

$$\mathbf{P}^T\mathbf{A}\mathbf{P}\alpha = -\mathbf{P}^T\mathbf{A}\lambda_\mathbf{0}\,, \tag{2}$$

where $\mathbf{A}$ is a block diagonal matrix, whose sub-matrices describe the local finite element stiffness matrices for associated single mesh cells, $\lambda_\mathbf{0}$ is a vector containing boundary data, α is the vector of global degrees of freedom (dof), and $\mathbf{P}$ is a very sparse incidence matrix mapping global to local degrees of freedom. The mathematical details of this are discussed in the full paper. Since $\mathbf{A}$ is positive definite and symmetric, solving equation 2 is equivalent to solving

$$(\lambda_\mathbf{0} + \mathbf{P}\alpha)^T\mathbf{A}(\lambda_\mathbf{0} + \mathbf{P}\alpha) \stackrel{!}{=} min\,. \tag{3}$$

Equation 3 is solved directly, using a direct search optimization algorithm, that changes only one global degree of freedom in each step. The used algorithm is a simple generating set search method as discussed in [4]. Further in the nonlinear case $\mathbf{A}$ depends on α which requires some additional care. This approach results in an algorithm with global convergence, even in many non-linear cases. The mathematical details and constraints are discussed in the full paper.

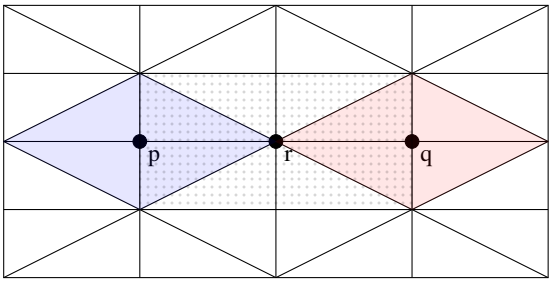

Fig. 1. Schematic sketch of a 2D Lagrange FEM mesh with affected cells and independent degrees of freedom. The affected cells of the dof associated with node p are blue, the ones associated with node q are red, and the ones associated with the node r are dotted. The set $\{p,q\}$ is a set of independent degrees of freedom, while the set $\{p,r,q\}$ is not.

2 Nodal Search

The nodal search algorithms exploits the fact, that a change in a global degree of freedom only affects a very local area of an approximation function build from finite elements. The basic idea is to find first, for every degree of freedom, the area where a change in the degree of freedom actually has an effect on the associated approximation function. This is done by finding the affected cells, as shown in Fig. 1, of every global degree of freedom.

Next, the algorithm determines a decomposition of all global degrees of freedom into a set of sets of independent degrees of freedom. Where independent means that the degrees of freedom have no affected cells in common, as shown in Fig. 1. A rigorous mathematical analysis of this decomposition approach is provided in the full paper. Last, an iteration over nodal search steps is done which terminates when the steplength becomes lower then a given threshold.

A nodal search step tries to improve the value of one node's dof. This leads to a formulation that is similar to a multiplicative Schwarz method and in

fact is, for first order finite elements, a multiplicative Schwarz method, which is shown in the full paper. Further in the first order case it is similar to the algorithm in [5] as well. The computation of a nodal search step obviously depends only on a few degrees of freedom. A nodal search step on node l improves an existing approximation described by α to a new better approximation α^{new} via $\alpha^{new} = \alpha + \theta_l \mathbf{e}_l$ where the nodal search step length θ_l can, for cellwise linear isotropic materials, be computed by

$$\theta_l = -\frac{\xi_l + \mathbf{v}_l^T \alpha}{\chi_l}, \tag{4}$$

where ξ_l, χ_l are real numbers and $\mathbf{v}_l$ is a sparse vector for all l. All values i.e. ξ_l, χ_l and v_l^T can be computed in parallel using $\lambda_0, \mathbf{P}$ and $\mathbf{A}$.

Further, the computation of all θ_l in a set of independent degrees of freedom is embarrassingly parallel, which is proven in the full paper. Thus, iterating through all sets of a decomposition into sets of independent degrees of freedom, yields a stepwise embarrassingly parallel algorithm.

3 Test Implementation and Results

The algorithm has been implemented with OpenMP parallelization. This test implementation uses second order tetrahedral elements, and an ad-hoc iterative mark and sweep algorithm to create the required decomposition of degrees of freedom into sets of independent degrees of freedom. The algorithm has been tested using the real world problem shown in Fig.2. The details of the model are described in [6].

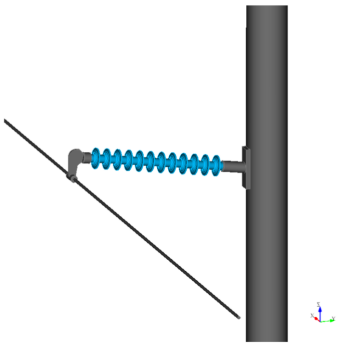

Fig. 2. A real world insulator model for testing, meshed using second order tetrahedrons with about 4.8 million nodes.

These results show, that the algorithm does not only work in theory, but in practice as well. The results show that the convergence is slow but secure even without preconditioning on ill conditioned systems. Its speed, of course, depends on the size of the problem and the amount of parallel processors. Thus a variant of this algorithm, e.g. extended by a multigrid scheme, might become a good default algorithm, on GPGPU systems, similarly to the approach in [7].

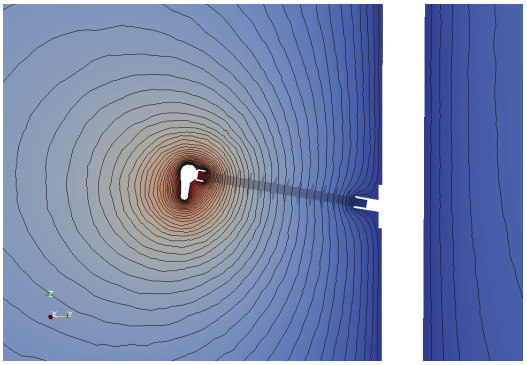

Fig. 3. A result for the geometry in Fig. 2 visualizing the electric potential by color and equipotential surfaces by black lines. Computation took about 10h on a 48core (2.8GHz AMD Opteron) machine.

4 Outlook

We have presented a new massively parallel algorithm for solving electrostatic problems. The full paper provides a rigorous mathematical treatment of the algorithm derivation and convergence with a lot more references. Further, the above and additions numerical results are discussed in detail, regarding convergence, speed and stability.

References

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