Efficient solvers for optimal control of eddy current problems with regularized state constraints

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Summary. This work is devoted to the robust and efficient solution of an optimal control problems for time-harmonic or time-periodic eddy current problems in the presence of pointwise state constraints imposed on the Fourier coefficients. For the discrete version of the linearized and reduced optimality systems of the Moreau-Yosida penalized minimization problems, condition number estimates of the preconditioned systems are provided. We show, that block-diagonal preconditioners can lead to parameter-robust and efficient solution strategies for these kind of problems.

1 Introduction

During recent years, the importance of solving optimization problems with constraints in form of partial differential equations has been growing. Usually, the partial differential equation is treated as a constraint, and the minimizing solution is determined by solving the corresponding optimality system. Typically, this approach leads to very bad condition systems of linear equations, and therefore the iterative solution of these kind of equations is a delicate issue.

In [2, 3] an optimal control problem with a simple time-periodic parabolic partial differential equation as the state equation is considered. The optimality system is discretized in terms of the harmonic balance finite element method, and parameter robust solvers are constructed for the resulting frequency domain equations. The aim of this work is to extend these ideas also to the eddy current optimal control problem, cf. [4, 5]. Therefore, we consider optimal control problems, where the partial differential equations is given by time-harmonic or time-periodic eddy current problems. Indeed, in the time-periodic setting, we establish the harmonic balance finite element method, in combination with efficient and robust solvers for the resulting frequency domain equations, as a powerful tool for solving optimal control problems in computational electromagnetics.

Furthermore, we include pointwise state constraints in our model, since they may be important to filter out undesired singularities in the solution of the eddy current problem.

2 Optimal control problem

We concentrate on the solution of the following optimal control problem:

$$\min_{(\mathbf{y^c},\mathbf{y^s},\mathbf{u^c},\mathbf{u^s})\in \mathbf{H_0}(\mathbf{curl})^2\times \mathbf{L_2}(\Omega)^2} J(\mathbf{y^c},\mathbf{y^s},\mathbf{u^c},\mathbf{u^s}), \tag{1}$$

subject to

$$\begin{cases} \mathbf{curl}(\nu\,\mathbf{curl}\,\mathbf{y^c}) + \omega\sigma\mathbf{y^s} = \mathbf{u^c}, & \text{in } \Omega,\\ \mathbf{curl}(\nu\,\mathbf{curl}\,\mathbf{y^s}) - \omega\sigma\mathbf{y^c} = \mathbf{u^s}, & \text{in } \Omega,\\ \mathbf{y^c}\times\mathbf{n} = \mathbf{y^s}\times\mathbf{n} = \mathbf{0}, & \text{on } \partial\Omega, \end{cases} \tag{2}$$

and to the pointwise state constraints

$$\mathbf{y_a^j}(\mathbf{x}) \le \mathbf{y^j}(\mathbf{x}) \le \mathbf{y_b^j}(\mathbf{x}), \quad \text{a.e. in } \Omega, j\in\{c,s\}. \tag{3}$$

The quadratic cost functional is given by

$$J(\mathbf{y^c},\mathbf{y^s},\mathbf{u^c},\mathbf{u^s}) := \frac{1}{2}\sum_{j\in\{c,s\}}\left[\|\mathbf{y^j}-\mathbf{y_d^j}\|_\mathbf{0}^2 + \lambda\|\mathbf{u^j}\|_\mathbf{0}^2\right].$$

The regularization parameter $\lambda > 0$, the model parameters σ, ν and ω, and $\mathbf{y_d^c},\mathbf{y_d^s},\mathbf{y_a^c},\mathbf{y_b^c},\mathbf{y_a^s},\mathbf{y_b^s}\in\mathbf{L_2}(\Omega)$ are given data.

Following [6], we use a Moreau-Yosida regularization, that penalizes the pointwise state constraints, i.e., we add the penalty term

$$\frac{1}{2\varepsilon}\sum_{j\in\{c,s\}}\|\max(\mathbf{0},\mathbf{y^j}-\mathbf{y_b^j})\|_\mathbf{0}^2 + \|\min(\mathbf{0},\mathbf{y^j}-\mathbf{y_a^j})\|_\mathbf{0}^2,$$

$\varepsilon > 0$, to the cost functional J. The resulting minimization can be solved by deriving the (reduced) optimality system. Due to the penalized state constraints, the optimality system becomes nonlinear. The nonlinearity can be dealt with in terms of a primal dual active set strategy, that is known to be equivalent to the semi-smooth Newton method [1]. At each Newton step, a two-fold saddle point problem has to be solved. Typically, the model parameters σ, ν and ω, the regularization parameters λ and ε, as well as the discretization parameter h, coming from the finite element approximation, impinge on the convergence of any iterative method applied to the unpreconditioned problem. Therefore, the aim of this paper is to present a preconditioning technique for the robust and efficient solution of the saddle point system at each Newton step.

3 Block-diagonal preconditioner

The finite element discretization of the penalized, linearized and reduced optimality system of (1)-(3), yields the linear system of equations

$$\mathscr{A}\mathbf{x} = \mathbf{b}, \tag{4}$$

where the system matrix $\mathscr{A}$ is given by

$$\mathscr{A} = \begin{pmatrix} \mathbf{M} + \frac{1}{\varepsilon}\mathbf{M}_{\mathscr{E}^\mathbf{c}} & \mathbf{0} & \mathbf{K}_\nu & -\mathbf{M}_{\omega,\sigma} \\ \mathbf{0} & \mathbf{M} + \frac{1}{\varepsilon}\mathbf{M}_{\mathscr{E}^\mathbf{s}} & \mathbf{M}_{\omega,\sigma} & \mathbf{K}_\nu \\ \mathbf{K}_\nu & \mathbf{M}_{\omega,\sigma} & -\frac{1}{\lambda}\mathbf{M} & \mathbf{0} \\ -\mathbf{M}_{\omega,\sigma} & \mathbf{K}_\nu & \mathbf{0} & -\frac{1}{\lambda}\mathbf{M}. \end{pmatrix}.$$

Here $\mathbf{K}_\nu$ corresponds to the stiffness matrix, $\mathbf{M}$ to the mass matrix, $\mathbf{M}_{\omega,\sigma}$ to a weighted mass matrix, and $\mathbf{M}_{\mathscr{E}^\mathbf{c}}$ and $\mathbf{M}_{\mathscr{E}^\mathbf{s}}$ to the mass matrices on the active sets $\mathscr{E}^c$ and $\mathscr{E}^s$, respectively. In order to solve (4), we follow the strategy used in [5] and construct a preconditioned MinRes solver. We propose and analyze the block-diagonal preconditioner

$$\mathscr{C} = \text{diag}\ (\sqrt{\lambda}\mathbf{E}, \sqrt{\lambda}\mathbf{E}, \frac{1}{\sqrt{\lambda}}\mathbf{E}, \frac{1}{\sqrt{\lambda}}\mathbf{E}), \tag{5}$$

where $\mathbf{E} = \mathbf{K}_\nu + \mathbf{M}_{\omega,\sigma} + \frac{1}{\sqrt{\lambda}}\mathbf{M}$. We show, that the condition number of the preconditioned system can be estimated by a constant, that is independent of the mesh size h, the regularization parameter λ, the model parameters σ, ν, and ω, as well as the active sets $\mathscr{E}^c$ and $\mathscr{E}^s$ from the primal dual active set strategy, i.e.,

$$\kappa(\mathscr{C}^{-1}\mathscr{A}) \leq c \neq c(\omega, \sigma, h, \lambda, \mathscr{E}^c, \mathscr{E}^s).$$

Therefore, the number of MinRes iterations required for reducing the initial error by some fixed factor $\delta \in (0,1)$ is independent of ω, σ, h, λ, $\mathscr{E}^c$, and $\mathscr{E}^s$. In practice, the diagonal blocks $\mathbf{E}$ of (5) are replaced by appropriate efficient and parameter robust preconditioners.

4 Time-periodic optimization

The presented solving technique provides a robust tool for solving optimal control problems with a time-harmonic eddy current problem as the state equation. Indeed, the theory can be extended to time-periodic optimal control problems of the form:

$$\min J(\mathbf{u}, \mathbf{y}) = \frac{1}{2}\int_0^T \|\mathbf{y} - \mathbf{y_d}\|_\mathbf{0}^2 \mathrm{dt} + \frac{\lambda}{2}\int_0^T \|\mathbf{u}\|_\mathbf{0}^2 \mathrm{dt},$$

subject to

$$\begin{cases} \sigma \dfrac{\partial \mathbf{y}}{\partial t} + \mathbf{curl}(\nu\, \mathbf{curl}\, \mathbf{y}) = \mathbf{u}, & \text{in } \Omega \times (0,T), \\ \mathbf{y} \times \mathbf{n} = \mathbf{0}, & \text{on } \partial\Omega \times (0,T), \\ \mathbf{y}(0) = \mathbf{y}(T), & \text{in } \Omega, \end{cases} \tag{6}$$

with state constraints associated to the Fourier coefficients of $\mathbf{y}$. Due to the periodic structure, a time approximation of the state $\mathbf{y}$ and the control $\mathbf{u}$ in terms of a truncated Fourier series can be used, i.e.,

$$\mathbf{y}(\mathbf{x}, t) = \sum_{k=0}^{N} \mathbf{y}_\mathbf{k}^\mathbf{c} \cos(k\omega t) + \mathbf{y}_\mathbf{k}^\mathbf{s} \sin(k\omega t).$$

Due to the linearity of (6), we obtain a decoupling of the frequency domain equations with respect to the individual modes $k = 0, \ldots, N$. For each mode, a linear system of equations, that obtains high structural similarities to (4) has to be solved. Hence, an efficient and parameter robust solver can be constructed in the same manner as done in the previous section. Indeed, this approach is an extension the harmonic balance finite element method to optimal control problems.

5 Conclusion

The method developed in this work shows great potential for solving both time-harmonic and time-periodic eddy current optimal control problems in an efficient and robust way.

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References

1. K. Ito and K. Kunisch. Semi-smooth Newton methods for state-constrained optimal control problems. *Systems Control Lett.*, 50(3):221–228, 2003.
2. M. Kollmann and M. Kolmbauer. A preconditioned minres solver for time-periodic parabolic optimal control problems. *Numer. Linear Algebra Appl.*, August 2011. accepted for publication in NLA.
3. M. Kollmann, M. Kolmbauer, U. Langer, M. Wolfmayr, and W. Zulehner. A finite element solver for a multiharmonic parabolic optimal control problem. NuMa-Report 2011-10, Institute of Computational Mathematics, Linz, December 2011.
4. M. Kolmbauer. Efficient solvers for multiharmonic eddy current optimal control problems with various constraints. NuMa-Report 2011-09, Institute of Computational Mathematics, Linz, November 2011.
5. M. Kolmbauer and U. Langer. A robust preconditioned MinRes solver for distributed time-periodic eddy current optimal control problems. NuMa-Report 2011-04, Institute of Computational Mathematics, Linz, May 2011.
6. I. Yousept. Optimal control of Maxwell's equations with regularized state constraints. *Computational Optimization and Applications*, pages 1–23, 2011.