Time Domain Models for Lossless Multiport Waveguide Structures in Impedance and Admittance Formulation based on Real Eigenmodes

Thomas Flisgen[1] and Ursula van Rienen[1]

Institut for General Electrical Engineering, University of Rostock, Albert-Einstein-Str. 2, 18059 Rostock, Germany.
thomas.flisgen@uni-rostock.de, ursula.van-rienen@uni-rostock.de

Summary. The creation of equivalent models of waveguide structures is a challenging task. This contribution revisites a method to establish systems of ordinary differential equations to model the transient dependency between port voltages and currents of the structure based on real eigenmodes. These equations can be derived either in an impedance or admittance formulation. The method is illustrated by the transfer properties of a rectangular waveguide. It is shown that the frequency domain impedance parameters of this waveguide, obtained by an eigenmode expansion, converge to the parameters which are commonly known from literature.

1 Introduction

The computation of transfer functions of lossless radio frequency (RF) multiport structures is a common issue related to scientific computing in electrical engineering. A large variety of methods is discussed in the literature to determine these transfer functions in frequency domain. However, frequency domain approaches fail, if field filling and defilling processes in RF structures [2, 4] or the response of these structures due to transient port stimuli are of interest. In this contribution a modal expansion is employed to transfer the electrodynamic wave equation with excitation to a coupled first order ordinary differential equation (ODE). This differential equation approximates the dependency of port quantities like voltages and currents in an explicit time domain formulation and is therefore referred to as lumped equivalent model of the structure. In this context it is remarkable that eigenmodes of the closed structure are suitable to expand the fields in a device with waveguide ports.

However, in contrast to previous publications on eigenmode expansions for multiport systems (see e.g. [1, 3, 5, 6]) this contribution performs the expansion using continuous fields and focusses on the creation of structure's equivalent ODE systems. In addition to theoretical derivations, the procedure is exemplified by means of a rectangular waveguide to obtain a time domain equivalent ODE model for this waveguide in an impedance formulation. It is shown that the frequency domain transfer function of the equivalent model converges to the well-known impedance parameters of the waveguide, if an infinite number of real eigenmodes is considered in the expansion.

2 Mathematical Modeling

For the derivation of waveguide structure's equivalent models Faraday's law of induction

$$\nabla \times \mathbf{E}(\mathbf{r},t) = -\mu \frac{\partial}{\partial t}\mathbf{H}(\mathbf{r},t) - \mathbf{J}_m(\mathbf{r},t) \qquad (1)$$

and Ampere's law

$$\nabla \times \mathbf{H}(\mathbf{r},t) = \varepsilon \frac{\partial}{\partial t}\mathbf{E}(\mathbf{r},t) + \mathbf{J}_e(\mathbf{r},t) \qquad (2)$$

are taken as a starting point. There, $\mathbf{E}(\mathbf{r},t)$ denotes the electric field strength, $\mathbf{H}(\mathbf{r},t)$ the magnetic field strength, ε a constant permittivity, μ a constant permeability, $\mathbf{J}_m(\mathbf{r},t)$ the magnetic and $\mathbf{J}_e(\mathbf{r},t)$ the electric current density.

2.1 Impedance Formulation

The impedance formulation describes the transient port voltages as a function of electric port currents. Therefore, the magnetic currents are set to zero in (1). Taking the curl of the resulting equation, using the electric fields to be free of any sources and replacing the curl of the magnetic fields by the r.h.s. of (2) leads to the wave equation with electric current excitation:

$$\Delta\mathbf{E}(\mathbf{r},t) - \varepsilon\mu \frac{\partial^2}{\partial t^2}\mathbf{E}(\mathbf{r},t) = \mu \frac{\partial}{\partial t}\mathbf{J}_e(\mathbf{r},t). \qquad (3)$$

The electric fields are expanded in terms of real eigenmodes $\tilde{\mathbf{E}}_\nu(\mathbf{r})$ and a transient weighting factor $x_\nu(t)$:

$$\mathbf{E}(\mathbf{r},t) = \sum_{\nu=1}^{N} \tilde{\mathbf{E}}_\nu(\mathbf{r}) x_\nu(t). \qquad (4)$$

Note, that the eigenmodes $\tilde{\mathbf{E}}_\nu(\mathbf{r})$ satisfy

$$\begin{aligned}
\Delta\tilde{\mathbf{E}}_\nu(\mathbf{r}) + \varepsilon\mu\tilde{\omega}_\nu^2\tilde{\mathbf{E}}_\nu(\mathbf{r}) &= \mathbf{0} \text{ on } \Omega, && (5)\\
\mathbf{n} \times \tilde{\mathbf{E}}_\nu(\mathbf{r}) &= \mathbf{0} \text{ on } \partial\Omega_{wall}, && (6)\\
\mathbf{n} \cdot \tilde{\mathbf{E}}_\nu(\mathbf{r}) &= \mathbf{0} \text{ on } \partial\Omega_{ports}, && (7)
\end{aligned}$$

with the resonant angular frequency $\tilde{\omega}_\nu$ of the ν-th mode. Condition (6) corresponds to perfect electric material on the boundary of the waveguide, whereas

(7) corresponds to perfect magnetic conducting material on the cross section of the waveguide ports. Replacing the transient electric fields in (3) by (4), exploiting the orthogonality of the eigenmodes

$$\iiint\limits_{\Omega} \tilde{\mathbf{E}}_\nu(\mathbf{r}) \cdot \tilde{\mathbf{E}}_n(\mathbf{r})\, \mathrm{d}\mathbf{r} = \begin{cases} 2W_\nu/\varepsilon, & \text{if } \nu = n, \\ 0, & \text{if } \nu \neq n, \end{cases} \tag{8}$$

and utilizing the fact that the excitation current density can be expressed as a superposition of impressed excitation current densities $\tilde{\mathbf{J}}_{port,m}(\mathbf{r})\, i_m(t)$ on the cross sections of all M waveguide ports

$$\mathbf{J}_e(\mathbf{r},t) = \sum_{m=1}^{M} \tilde{\mathbf{J}}_{port,m}(\mathbf{r})\, i_m(t) \tag{9}$$

leads to the ordinary differential equation

$$2W_\nu \left(\tilde{\omega}_\nu^2 x_\nu(t) + \frac{\partial^2 x_\nu(t)}{\partial t^2} \right) = \frac{\partial}{\partial t} \sum_{m=1}^{M} f_{\nu,m}\, i_m(t). \tag{10}$$

Here W_ν is defined as the energy stored in the ν-th mode, $i_m(t)$ the modal current of the m-th port and

$$f_{\nu,m} = \iiint\limits_{\Omega} \tilde{\mathbf{E}}_\nu(\mathbf{r}) \cdot \tilde{\mathbf{J}}_{port,m}(\mathbf{r})\, \mathrm{d}\mathbf{r}. \tag{11}$$

Equation (10) can be expressed for all N considered eigenmodes and all M ports as the state equation

$$\frac{\partial}{\partial t} \underbrace{\begin{pmatrix} \hat{x}_1(t) \\ x_1(t) \\ \vdots \\ \hat{x}_N(t) \\ x_N(t) \end{pmatrix}}_{\mathbf{x}(t)} = \underbrace{\begin{pmatrix} 0 & 1 \dots & 0 & 0 \\ -\tilde{\omega}_1^2 & 0 \dots & 0 & 0 \\ \vdots & \vdots \ddots & \vdots & \vdots \\ 0 & 0 \dots & 0 & 1 \\ 0 & 0 \dots & -\tilde{\omega}_N^2 & 0 \end{pmatrix}}_{\mathbf{A}_z} \underbrace{\begin{pmatrix} \hat{x}_1(t) \\ x_1(t) \\ \vdots \\ \hat{x}_N(t) \\ x_N(t) \end{pmatrix}}_{\mathbf{x}(t)}$$
$$+ \frac{1}{2} \underbrace{\begin{pmatrix} 0 & 0 & \dots & 0 \\ \frac{f_{1,1}}{W_1} & \frac{f_{1,2}}{W_1} & \dots & \frac{f_{1,M}}{W_1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \\ \frac{f_{N,1}}{W_N} & \frac{f_{N,2}}{W_N} & \dots & \frac{f_{N,M}}{W_N} \end{pmatrix}}_{\mathbf{B}_z} \underbrace{\begin{pmatrix} i_1(t) \\ i_2(t) \\ \vdots \\ i_M(t) \end{pmatrix}}_{\mathbf{i}(t)}. \tag{12}$$

The dependency of the voltages at the ports and the inner states is described via the output equation

$$\underbrace{\begin{pmatrix} v_1(t) \\ v_2(t) \\ \vdots \\ v_M(t) \end{pmatrix}}_{\mathbf{v}(t)} = \underbrace{\begin{pmatrix} 0 & f_{1,1} & \dots & 0 & f_{N,1} \\ 0 & f_{1,2} & \dots & 0 & f_{N,2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & f_{1,M} & \dots & 0 & f_{N,M} \end{pmatrix}}_{\mathbf{C}_z} \underbrace{\begin{pmatrix} \hat{x}_1(t) \\ x_1(t) \\ \vdots \\ \hat{x}_N(t) \\ x_N(t) \end{pmatrix}}_{\mathbf{x}(t)}. \tag{13}$$

Note that the state space system (12) and (13) with the matrices $\{\mathbf{A}_z, \mathbf{B}_z, \mathbf{C}_z\}$ is an equivalent ODE model of the multiport waveguide structure in an impedance formulation, since it describes the transient port voltages as a function of transient port currents.

2.2 Admittance Formulation

The derivation of the state space systems in an admittance formulation is similar to the procedure proposed in section 2.1. However, for the admittance formulation magnetic currents are used to excite the structure, the expansion is performed using magnetic fields and perfect electric conducting boundary conditions are chosen on the port cross sections for the eigenmodes.

3 Application Example and Convergence

A homogeneously filled rectangular waveguide with constant cross section carrying a TE_{10} mode is chosen as an example. This structure is well suited for validation and demonstration purposes as the eigenvalue problem (5) - (7) can be solved analytically for this geometry. As a central result of this contribution it is shown, that the frequency domain transfer function of the state space system (12) and (13) converges to the analytically known impedance matrix $\mathbf{Z}(\mathrm{i}\omega)$ of the waveguide, if an infinite number of eigenmodes is considered in the modal expansion:

$$\lim_{N\to\infty} \mathbf{C}_z \left(\mathrm{i}\omega \mathbf{I} - \mathbf{A}_z \right)^{-1} \mathbf{B}_z = \mathbf{Z}(\mathrm{i}\omega). \tag{14}$$

Here $\mathbf{I} \in \mathbb{R}^{N\times N}$ denotes the identity matrix .

4 Summary and Conclusions

This work illustrates the derivation of ODE models in an impedance and admittance formulation for lossless RF structures based on real eigenmodes. The method is exemplified by employing a simple waveguide. Furthermore, the convergence of the method is discussed for this test example.

References

1. J.R. Brauer and G.C. Brauer. Microwave filter analysis using a new 3-D finite-element modal frequency method. *IEEE Trans. Microw. Theory Tech.*, 45(5):810 – 818, May 1997.
2. M. Dohlus, H.-W.Glock, D. Hecht, and U. van Rienen. Filling and beam loading in TESLA superstructures. TESLA-REPORT 98-14, 1998.
3. M. Dohlus, R. Schuhmann, and T. Weiland. Calculation of frequency domain parameters using 3D eigensolutions. *International Journal of Numerical Modelling*, 12:41 – 48, Januar 1999.
4. H.-W. Glock, M. Kurz, P. Hülsmann, and H. Klein. Rise time of the amplitudes of time harmonic fields in multicell cavities. In *Proc. of the Particle Accelerator Conference 1993*, volume 1, pages 623 – 625, May 1993.
5. J. C. Slater. Microwave electronics. *Rev. Mod. Phys.*, 18:441–512, Oct 1946.
6. T. Wittig, R. Schuhmann, and T. Weiland. Model order reduction for large systems in computational electromagnetics. *LinAlgApp*, 415(2 - 3):499 – 530, 2006.